Supersymmetric localization with dynamical gravitons

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Introduction and motivation

In supergravity, as in any gauge theory, BRST quantization provides a convenient and consistent setting for integrating over quantum fluctuations. However, in the presence of a background/boundary there is also a rigid symmetry group of isometries which is a subgroup of the local gauge group. How do the isometries act on the quantum fields and is there a charge associated with them?

Question: How can one consistently deal with these two different but yet closely related symmetries?

This is an essential issue that confronts the application of localization in supergravity. When one is dealing with different independent symmetries there is no problem. For instance, for a supersymmetric gauge theory one can combine the BRST charge with a rigid supersymmetry charge and define equivariant cohomology. But when considering the full supergravity then all these invariances are contained in one common irreducible gauge algebra.

Nekrasov, 2003
Pestun, 2012
Pestun et al., 2016

The above observations are important because there are applications in physics where one should not freeze the space-time metric, so that the gravitons will still represent dynamical degrees of freedom over which one should integrate in the functional integral. Until recently it was not entirely clear how to carry out such a calculation consistently.

One such application concerns the calculation of the entropy of BPS black holes. For these (charged, extremal) black holes the near-horizon geometry equals $\mathrm{AdS}_2 \times S^2$. Because of the supersymmetry enhancement at the horizon, the horizon values of the fields and the size of the horizon area are determined by the charges. The entropy can then be obtained from the horizon data by making use of the Bekenstein-Hawking area law, or, when the action contains also higher-derivative couplings, by using the Wald entropy formula.

Ferrara, Kallosh, Strominger, 1996
Cardoso, dW, Mohaupt, 1999, 2000

Obviously, this approach did not take into account the quantum fluctuations of the supergravity fields in the AdS space. This can be accomplished by using Sen's quantum entropy function which is based on the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. That means that one has to evaluate a path integral over all fields living in a space-time with a boundary, where one integrates over the super-gravitational degrees of freedom.

Sen, 2008

In recent years there have been several calculations of the quantum entropy function based on localization, but still without including all possible quantum fluctuations.

Dabholkar, Gomes, Murthy, 2011, 2013
Murthy, Reys, 2013 , 2015
Hristov, Lodato, Reys, 2018, 2019

In this talk I will discuss recent progress on this issue and explain how to apply localization for spaces with a boundary which, at the same time, can deal with the integration over all fluctuating supergravity modes.

dW, Reys, Murthy, 1806.03690

Meanwhile this framework has already been tested in an actual calculation.

Jeon, Murthy, 1806.04479

What is localization?

Localization is a technique by which, under certain conditions, an integral can be exactly written as an integral over a lower-dimensional subspace, or sometimes as a sum over contributions of fixed points. This technique was originally developed by mathematicians in the twentieth century.

Duistermaat, Heckman, Berline, Vergne, Atiyah, Bott, 1982 - 1984

The idea of localization can also be applied to the path integrals that appear in quantum field theory. Usually this is done in the context of supersymmetric field theories. When (supersymmetric) localization is applicable, the infinite-dimensional path integral can be expressed in terms of an expansion about a restricted field configuration that is known as the localization manifold.

Note that this localization manifold has no intrinsic physical significance. It is actually induced by a special deformation of the action which is chosen such that, in a convenient limit, the full path integral is determined by the action taken on the localization manifold while only including the semiclassical corrections associated with quantum fluctuations about this manifold.

Content

1 - Quantization of the gauge theory (in our case supergravity). This requires the introduction of BRST cohomology.

2 - Incorporating the boundary data. Here we will use a background field splitting that is suitable for theories with a soft gauge algebra.

3 - For localization one needs an equivariant cohomology. What is the connection?

4 - For localization one also needs to introduce a deformation which in a special limit leads to a suitable localization manifold.

BRST cohomology

We consider a generic gauge theory with transformation

$$\delta\phi^i = R^i{}_\alpha(\phi)\,\xi^\alpha$$

where $R_\alpha{}^i(\phi)$ *depends on fields and may contain space-time derivatives, and the parameters* $\xi^\alpha(x)$ *are functions of the space-time coordinates.*

These transformations close under commutation,

$$\delta(\xi_1)\,\delta(\xi_2) - \delta(\xi_2)\,\delta(\xi_1) = \delta(\xi_3)$$

with $\xi_3{}^\alpha = f_{\beta\gamma}{}^\alpha\,\xi_1{}^\beta\,\xi_2{}^\gamma$ *and structure 'constants'* $f_{\alpha\beta}{}^\gamma$ *that may depend on the fields, so that the algebra is 'soft'. This implies:*

Closure: $R^j{}_{[\alpha}\,\partial_j R^i{}_{\beta]} = \frac{1}{2} f_{\alpha\beta}{}^\gamma\,R^i{}_\gamma$*, and correspondingly the*

Jacobi identity: $f_{[\alpha\beta}{}^\delta\,f_{\gamma]\delta}{}^\epsilon + \boxed{R^j{}_{[\alpha}\,\partial_j f_{\beta\gamma]}{}^\epsilon} = 0$

soft algebra

For simplicity we restrict ourselves to bosonic transformations but we are actually interested in transformations with both commuting and anti-commuting transformation parameters!

The corresponding BRST *transformations*

$$\delta_{\rm brst}\phi^i = R(\phi)^i{}_\alpha\,\Lambda\,c^\alpha \qquad \delta_{\rm brst}c^\alpha = \tfrac{1}{2}\,f_{\beta\gamma}{}^\alpha\,c^\beta\,\Lambda\,c^\gamma$$

are nilpotent

$$\delta_{\rm brst}{}^2\,\phi^i = 0 \qquad \delta_{\rm brst}{}^2\,c^\alpha = 0$$

The path integral requires gauge-fixing. Therefore one includes

$$\mathcal{L}^{\rm g.f.} = \partial_\Lambda\,\delta_{\rm brst}\big[b_\alpha\,F(\phi)^\alpha\big] = B_\alpha\,F(\phi)^\alpha - b_\alpha\,R(\phi)^j{}_\beta\,c^\beta\,\partial_j F(\phi)^\alpha$$

where we have assumed the following transformation rules

$$\delta_{\rm brst}b_\alpha = \Lambda\,B_\alpha \qquad \delta_{\rm brst}B_\alpha = 0$$

The fields B_α are Lagrange multipliers that impose the gauge conditions

$$F(\phi)_\alpha = 0$$

The fields b_α are known as the anti-ghost fields.

The statistics of the ghost and anti-ghosts is opposite to the statistics of the corresponding gauge fields and transformation parameters.

Background field split

To be able to describe boundary conditions we make use of a background field split:

$$\phi^i = \mathring{\phi}^{\,i} + \widetilde{\phi}^{\,i}$$

The boundary fields $\mathring{\phi}^{\,i}$ will be fixed at the boundary and they are continued into the bulk. The precise continuation is not important. The quantum fields $\widetilde{\phi}^{\,i}$ will be integrated over in the functional integral and they will vanish at the boundary.

Background transformations

$$\mathring{\delta}\mathring{\phi}^i = R(\mathring{\phi})^i{}_\alpha\,\mathring{\xi}^{\,\alpha} \qquad\qquad \mathring{\delta}\widetilde{\phi}^{\,i} = \Delta R(\mathring{\phi},\widetilde{\phi})^i{}_\alpha\,\mathring{\xi}^{\,\alpha}$$

where $\Delta R(\mathring{\phi},\widetilde{\phi})^i{}_\alpha \equiv R(\mathring{\phi}+\widetilde{\phi})^i{}_\alpha - R(\mathring{\phi})^i{}_\alpha$

Quantum transformations

$$\tilde{\delta}\mathring{\phi}^{\,i} = 0 \qquad\qquad \tilde{\delta}\widetilde{\phi}^{\,i} = R(\mathring{\phi}+\widetilde{\phi})^i{}_\alpha\,\xi^\alpha$$

Algebra of background and quantum transformations

$$\big[\tilde{\delta}(\xi_1)\,\tilde{\delta}(\xi_2) - \big(1 \leftrightarrow 2\big)\big]\mathring{\phi}^i = 0\,,$$

$$\big[\mathring{\delta}(\mathring{\xi})\,\tilde{\delta}(\xi) - \tilde{\delta}(\xi)\,\mathring{\delta}(\mathring{\xi})\big]\mathring{\phi}^i = 0\,,$$

$$\big[\mathring{\delta}(\mathring{\xi}_1)\,\mathring{\delta}(\mathring{\xi}_2) - \big(1 \leftrightarrow 2\big)\big]\mathring{\phi}^i = f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\,\mathring{\xi}_1{}^{\alpha}\,\mathring{\xi}_2{}^{\beta}\,R(\mathring{\phi})^i{}_{\gamma}$$

$$\big[\tilde{\delta}(\xi_1)\,\tilde{\delta}(\xi_2) - \big(1 \leftrightarrow 2\big)\big]\widetilde{\phi}^i = f(\mathring{\phi} + \widetilde{\phi})_{\alpha\beta}{}^{\gamma}\,\xi_1{}^{\alpha}\,\xi_2{}^{\beta}\,R(\mathring{\phi} + \widetilde{\phi})^i{}_{\gamma}$$

$$\big[\mathring{\delta}(\mathring{\xi})\,\tilde{\delta}(\xi) - \tilde{\delta}(\xi)\,\mathring{\delta}(\mathring{\xi})\big]\widetilde{\phi}^i = f(\mathring{\phi} + \widetilde{\phi})_{\alpha\beta}{}^{\gamma}\,\mathring{\xi}^{\alpha}\,\xi^{\beta}\,R(\mathring{\phi} + \widetilde{\phi})^i{}_{\gamma}$$

$$\begin{aligned}\big[\mathring{\delta}(\mathring{\xi}_1)\,\mathring{\delta}(\mathring{\xi}_2) - \big(1 \leftrightarrow 2\big)\big]\widetilde{\phi}^i &= f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\,\mathring{\xi}_1{}^{\alpha}\,\mathring{\xi}_2{}^{\beta}\,\Delta R(\mathring{\phi}, \widetilde{\phi})^i{}_{\gamma} \\ &\quad + \boxed{\big[f(\phi) - f(\mathring{\phi})\big]_{\alpha\beta}{}^{\gamma}}\,\mathring{\xi}_1{}^{\alpha}\,\mathring{\xi}_2{}^{\beta}\,R(\mathring{\phi} + \widetilde{\phi})^i{}_{\gamma}\end{aligned}$$

soft algebra

$$[\,\mathring{\delta}\,,\mathring{\delta}\,] = \mathring{\delta} + \widetilde{\delta} \qquad [\,\mathring{\delta}\,,\widetilde{\delta}\,] = \widetilde{\delta} \qquad [\,\widetilde{\delta}\,,\widetilde{\delta}\,] = \widetilde{\delta}$$

The background transformations do not form a closed subalgebra!

BRST transformations and the action

$$\delta_{\rm brst}\mathring{\phi}^{\,i} = R(\mathring{\phi})^i{}_\alpha\,\Lambda\,\mathring{c}^{\,\alpha}$$

$$\delta_{\rm brst}\widetilde{\phi}^i = R(\mathring{\phi}+\widetilde{\phi})^i{}_\alpha\,\Lambda\,(c^\alpha+\mathring{c}^{\,\alpha}) - \boxed{R(\mathring{\phi})^i{}_\alpha}\,\Lambda\,\mathring{c}^{\,\alpha}$$

$$\delta_{\rm brst}\,\mathring{c}^{\,\gamma} = \tfrac{1}{2}\,f(\mathring{\phi})_{\alpha\beta}{}^\gamma\,\mathring{c}^{\,\alpha}\Lambda\,\mathring{c}^{\,\beta}$$

$$\begin{aligned}\delta_{\rm brst}\,c^{\,\gamma} &= \tfrac{1}{2}\,f(\phi)_{\alpha\beta}{}^\gamma\,c^\alpha\Lambda\,c^\beta + f(\phi)_{\alpha\beta}{}^\gamma\,\mathring{c}^{\,\alpha}\Lambda\,c^\beta + \tfrac{1}{2}\,\big[f(\phi)-f(\mathring{\phi})\big]_{\alpha\beta}{}^\gamma\,\mathring{c}^{\,\alpha}\,\Lambda\,\mathring{c}^{\,\beta}\\ &= \tfrac{1}{2}\,f(\phi)_{\alpha\beta}{}^\gamma\,(c+\mathring{c})^\alpha\Lambda\,(c+\mathring{c})^\beta - \tfrac{1}{2}\,\boxed{f(\mathring{\phi})_{\alpha\beta}{}^\gamma}\,\mathring{c}^{\,\alpha}\,\Lambda\,\mathring{c}^{\,\beta}\end{aligned}$$

soft algebra

$$\delta_{\rm brst}\,b_\alpha = \Lambda\,B_\alpha \qquad\qquad \delta_{\rm brst}B_\alpha = 0$$

$$\begin{aligned}S_{\rm brst}[\widetilde{\phi}^{\,i},c^\alpha,b_\alpha,B_\alpha;\mathring{\phi}^{\,i},\mathring{c}^{\,\alpha}] = \int {\rm d}^nx\,\Big[&\mathcal{L}^{\rm class}(\mathring{\phi}+\widetilde{\phi}) + B_\alpha\,F(\mathring{\phi},\widetilde{\phi})^\alpha\\ &-(-)^{\epsilon_\alpha+\epsilon_\beta+\epsilon_j}\,b_\alpha\,R(\mathring{\phi}+\widetilde{\phi})^j{}_\beta\,(c+\mathring{c})^{\,\beta}\,\tilde{\partial}_jF(\mathring{\phi},\widetilde{\phi})^\alpha\\ &-(-)^{\epsilon_\alpha+\epsilon_\beta+\epsilon_j}\,b_\alpha\,R(\mathring{\phi})^j{}_\beta\,\mathring{c}^{\,\beta}\,(\mathring{\partial}-\tilde{\partial})_jF(\mathring{\phi},\widetilde{\phi})^\alpha\Big]\end{aligned}$$

$$\boxed{\delta_{\rm brst}S_{\rm brst} = 0} \qquad\qquad \boxed{{\delta_{\rm brst}}^2 = 0}$$

The object of study will be the functional integral,

$$Z[\mathring{\phi}] = \int \mathcal{D}\widetilde{\phi}^{\,i}\,\mathcal{D}c^{\alpha}\,\mathcal{D}b_{\alpha}\,\mathcal{D}B_{\alpha}\;\exp\Big[S_{\rm brst}[\widetilde{\phi}^{\,i}, c^{\alpha}, b_{\alpha}, B_{\alpha}; \mathring{\phi}^{\,i}, \mathring{c}^{\,\alpha}]\Big]$$

It depends only the background fields and not on the background ghosts and satisfies (note that the integration measure is invariant).

$$\boxed{\delta_{\rm brst} Z[\mathring{\phi}] = \frac{\partial Z[\mathring{\phi}]}{\partial \mathring{\phi}^{\,i}}\, R(\mathring{\phi})^{i}{}_{\alpha}\, \mathring{c}^{\,\alpha}}$$

The functional integral is independ of the gauge condition. This follows from the Ward identities associated with BRST *invariance.*
The gauge independence will remain intact when one includes a BRST-*exact deformation of the form* $\delta_{\rm brst}\,\mathcal{V}(\mathring{\phi}, \widetilde{\phi})$.

Another implication of the Ward identities is that the Lagrange multiplier fields B_{α} *must have a vanishing expectation value.*

Reminder: we have everywhere assumed that the BRST transformations close off shell!

Towards equivariant cohomology

The background fields $\mathring{\phi}^{\,i}$ will be fixed by physical considerations and will be invariant under an isometry group that is a subgroup of the full group of background transformations. In the continuation of the backgrond fields into the bulk, the isometry group can remain manifest. This implies that the BRST *variations of the background should vanish, i.e.*

$$\delta_{\rm brst}\mathring{\phi}^{\,i} = R(\mathring{\phi})^i{}_\alpha\,\Lambda\,\mathring{c}^{\,\alpha} = 0$$

Consequently all the background ghosts should vanish with the exception of those that parametrize the isometry group, Remarkably enough the transformation of the background ghosts remains unchanged and does not require additional constraints. Note that

$$\delta_{\rm brst}\mathring{c}^{\,\alpha} = \tfrac{1}{2}\,f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}\,\mathring{c}^{\,\beta}\Lambda\,\mathring{c}^{\,\gamma}$$

does not vanish, but is now restricted to the isometry sub-algebra

With these restrictions the BRST *transformations are still nilpotent and the functional integral is* BRST *invariant!*

Subsequently we consider a deformation where all background fields and ghosts remain invariant. This leads to equivariant cohomology.

Deform the BRST algebra

$$\delta_{\text{eq}}\, \mathring{\phi}^{\,i} = 0 \qquad \boxed{\delta_{\text{eq}}\, \mathring{c}^{\,\gamma} = 0}$$

truncation

$$\delta_{\text{eq}}\, \widetilde{\phi}^{\,i} = R(\mathring{\phi} + \widetilde{\phi})^i{}_\alpha\, \Lambda\, (c^\alpha + \mathring{c}^{\,\alpha})$$

$$\delta_{\text{eq}}\, c^{\,\alpha} = \tfrac{1}{2}\, f(\phi)_{\beta\gamma}{}^\alpha\, (c + \mathring{c})^\beta \Lambda\, (c + \mathring{c})^\gamma - \tfrac{1}{2}\, f(\mathring{\phi})_{\beta\gamma}{}^\alpha\, \mathring{c}^{\,\beta}\, \Lambda\, \mathring{c}^{\,\gamma}$$

$$\delta_{\text{eq}}\, B_\alpha = \boxed{\tfrac{1}{2}\, f(\mathring{\phi})_{\delta\varepsilon}{}^\beta\, \mathring{c}^{\,\delta}\, \Lambda\, \mathring{c}^{\,\varepsilon}\, f(\mathring{\phi})_{\alpha\beta}{}^\gamma\, b_\gamma}$$

changed

$$\delta_{\text{eq}}\, b_\alpha = \Lambda\, B_\alpha$$

Due to the deformation the transformations are no longer nilpotent!
Instead we obtain an equivariant map

$$\boxed{{\delta_{\text{eq}}}^2 = \delta_{\mathring{\xi}} \qquad [\delta_{\text{eq}}\, , \delta_{\mathring{\xi}}] = 0}$$

$\mathring{\xi}^\alpha$ *takes its values in the isometry algebra and is quadratic in the background ghosts.*

The background ghosts now play an ancillary role as the parameters of the isometry transformations, which act both on the background and on the quantum fields!

$\delta_{\mathring{\xi}}$ *acts only on the quantum fields:*

$$\begin{aligned}
\delta_{\mathring{\xi}}\,\widetilde{\phi}^{\,i} &= R(\mathring{\phi}+\widetilde{\phi})^{i}{}_{\alpha}\,\mathring{\xi}^{\alpha}\\
\delta_{\mathring{\xi}}\,c^{\,\alpha} &= f(\mathring{\phi}+\widetilde{\phi})_{\beta\gamma}{}^{\alpha}\,(c+\mathring{c})^{\beta}\,\mathring{\xi}^{\gamma}\\
\delta_{\mathring{\xi}}\,b_{\alpha} &= \mathring{\xi}^{\beta}\,f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\,b_{\gamma}\\
\delta_{\mathring{\xi}}\,B_{\alpha} &= \mathring{\xi}^{\beta}\,f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\,B_{\gamma}
\end{aligned}$$

This is an important feature. Note that these variations vanish at the boundary (where the quantum fields themselves are required to vanish). The same is true for the equivariant variations.

Let us now return to the functional integral and consider the replacement of the BRST *variations by equivariant variations. Hence*

$$S_{\rm eq}[\widetilde{\phi}^{\,i},c^{\alpha},b_{\alpha},B_{\alpha};\mathring{\phi}^{\,i},\mathring{c}^{\,\alpha}] = \int \mathrm{d}^n x\,\Big[\mathcal{L}^{\rm class}(\mathring{\phi}+\widetilde{\phi}) + \partial_{\Lambda}\,\delta_{\rm eq}\big[b_{\alpha}\,F(\mathring{\phi},\widetilde{\phi})^{\alpha}\big]\Big]$$

which turns out to be identical to the BRST *expression for the action! To see whether the corresponding functional integral is now also invariant under the equivariant variations is a bit more involved.*

One easily derives

$$\delta_{\rm eq}\, S_{\rm eq} = \delta_{\mathring{\xi}} \int \mathrm{d}^n x\, \big[b_\alpha\, F(\mathring{\phi}, \widetilde{\phi})^\alpha \big]$$

where we used that the boundary is invariant under $\delta_{\rm eq}$. The right-hand side will in principle contribute when evaluating the equivariant variation on the functional integral.

However, this cancellation can still be realized by assuming that the background isometry $\delta_{\mathring{\xi}}$ is compact. This requires to make a special selection for the background ghosts.

Compactness of the manifold on which the theory is defined is also a requirement.

On the basis of this assumption it follows that the functional integral must be independent of the gauge condition. Likewise one can introduce $\delta_{\rm eq}$-exact deformations into the action without affecting the the invariance of the functional integral and the independence of the gauge condition.

To evaluate the functional integral exactly one can in principle apply localization by making use of the formalism described so far.

Localization of the functional integral

The first step is to introduce a deformation into the action $S_{\rm eq}$, equal to

$$\lambda\,\delta_{\rm eq}\,\mathcal{V}$$

which satisfies ${\delta_{\rm eq}}^2\,\mathcal{V}=0$. Furthermore

$$\frac{d}{d\lambda}Z[\mathring{\phi};\lambda]=\int\mathcal{D}\widetilde{\phi}^{\,i}\,\mathcal{D}c^{\alpha}\,\mathcal{D}b_{\alpha}\,\mathcal{D}B_{\alpha}\;\delta_{\rm eq}\left[\mathcal{V}\,\exp[S_{\rm eq}+\lambda\,\delta_{\rm eq}\,\mathcal{V}\,]\right]$$

Assuming that $\delta_{\rm eq}$ can be represented as a differential operator in the field configuration space, it follows that (super-Stokes theorem),

$$\frac{d}{d\lambda}Z[\mathring{\phi};\lambda]=0$$

Schwarz, Zaboronsky, 1997

An immediate consequence is that we can take the limit of infinite λ, so the integral localizes on the critical points of the deformation $\delta_{\rm eq}\,\mathcal{V}$.

More qualitatively: when this is done correctly the result is given by the value of the Lagrangian on the localization manifold modified by semiclassical corrections.

A convenient (standard) choice for the deformation is

$$\mathcal{V} = \int \mathrm{d}^n x \sqrt{\mathring{g}(x)} \sum_{\bar{\imath}} \bar{\psi}_{\bar{\imath}} \, \delta_{\mathrm{eq}} \, \psi^{\bar{\imath}}$$

where $\psi^{\bar{\imath}}$ *denote all the fermonic quantum fields that do not overlap with the fermonic gauge-fixing terms.*

Remember that we required ${\delta_{\mathrm{eq}}}^2 \mathcal{V} = 0$, *which is indeed satisfied in view of*

$$\delta_{\mathring{\xi}} \mathcal{V} = \delta_{\mathring{\xi}} \int \mathrm{d}^n x \sum_{\bar{\imath}} \sqrt{\mathring{g}} \, \overline{\psi}_{\bar{\imath}} \, \delta_{\mathrm{eq}} \psi^{\bar{\imath}} = 0$$

In the limit $\lambda \to \infty$ *the critical points of the deformation*

$$\lambda \, \delta_{\mathrm{eq}} \mathcal{V} = \lambda \int \mathrm{d}^n x \sqrt{\mathring{g}} \sum_{\bar{\imath}} \left[\delta_{\mathrm{eq}} \overline{\psi}_{\bar{\imath}} \, \delta_{\mathrm{eq}} \psi^{\bar{\imath}} - \overline{\psi}_{\bar{\imath}} \, {\delta_{\mathrm{eq}}}^2 \psi^{\bar{\imath}} \right]$$

must satisfy $\delta_{\mathrm{eq}} \psi^{\bar{\imath}} = 0$.

Provided that the bar on the fermions defines an appropriate conjugation so that the quadratic form is positive definite. Furthermore we assumed that the critical locus is bosonic.

Under these assumptions the localization manifold is defined by

$$\boxed{\mathcal{M} = \left\{ \delta_{\rm eq}\psi^{\bar\imath} = 0 \ \ \text{for all fermions } \psi^{\bar\imath} \in \widetilde{\phi}^{\,i} \,\big/ F(\mathring{\phi},\widetilde{\phi})^\alpha = 0 \,\right\} \equiv \{t_a\}}$$

where the parameters t_a are appropriately chosen coordinates on the solution set $\mathcal{M}$.

The localization manifold involves, in principle, the bosonic quantum fields of the original supergravity and the bosonic ghosts associated with the fermonic gauge transformations. The bosonic multiplier fields and the anti-ghosts will eventually mix, but their role is somewhat different.

The relevant action takes the following form:

$$S(\lambda) = S^{\rm class}[\mathring{\phi} + \widetilde{\phi}] + \int \mathrm{d}^n x \left[B_\alpha F(\mathring{\phi},\widetilde{\phi})^\alpha + (-)^{\epsilon_\alpha} b_\alpha\, \delta_{\rm eq} F(\mathring{\phi},\widetilde{\phi})^\alpha \,\right]$$

$$+ \lambda \int \mathrm{d}^n x \sum_{\bar\imath} \sqrt{\mathring{g}}\, \left[\delta_{\rm eq}\overline{\psi}_{\bar\imath}\, \delta_{\rm eq}\psi^{\bar\imath} - \overline{\psi}_{\bar\imath}\, {\delta_{\rm eq}}^2 \psi^{\bar\imath} \,\right]$$

There is a balanced set of quantum fields with equal number of fermions and bosons (provided one takes account of the gauge conditions).

To illustrate what happen in the limit $\lambda \to \infty$ it is convenient to introduce the rescalings

$$\widetilde{\phi}^{\,i} = \widetilde{\phi}^{\,i}(t)\big|_{\mathcal{M}} + \frac{1}{\sqrt{\lambda}}\,\widetilde{\phi}^{\,i\,\prime} \qquad c^{\alpha} = c^{\,\alpha}(t)\big|_{\mathcal{M}} + \frac{1}{\sqrt{\lambda}}\,c^{\alpha\,\prime}$$

$$b_{\alpha} = \sqrt{\lambda}\,b_{\alpha}{}' \qquad B^{\alpha} = \sqrt{\lambda}\,B^{\alpha\prime}$$

The first contribution is the classical action evaluated on the localization manifold. Note that the measure of the functional integral is insensitive to these rescalings owing to the balance between fermionic and bosonic fields.

The second contribution originates from the integral over the fluctuations about the localization manifold.

$$Z[\mathring{\phi}] = \int_{\mathcal{M}} \mu(t)\,\mathrm{d}t_a\, \exp\left[S^{\rm class}[\mathring{\phi}, \mathring{c}\,; t_a]\right]\, Z_{\text{1-loop}}[\mathring{\phi}, \mathring{c}\,; t_a]$$

measure associated with the embedding of
the localisations manifold

Then the semiclassical contribution to the path integral equals

$$Z_{\text{1-loop}}[\mathring{\phi}, \mathring{c}\,; t_a] = \int \mathcal{D}(\widetilde{\phi}^{\,i\,\prime})\,\mathcal{D}(c^{\alpha\,\prime})\,\mathcal{D}(b_\alpha{}')\,\mathcal{D}(B_\alpha{}')$$

$$\times \exp\Big[\delta_{\text{eq}}\big[\mathcal{V} + b_\alpha{}' F(\mathring{\phi}\,; t_a, \widetilde{\phi}\,')^\alpha\,\big]\Big]\Big|_{\text{quad.}}$$

where the terms quadratic in the quantum fields decompose according to

$$\delta_{\text{eq}}\mathcal{V} = \int \mathrm{d}^n x\,\sqrt{\mathring{g}}\,\sum_{\bar{\imath}}\,\Big[\delta_{\text{eq}}\overline{\psi}_{\bar{\imath}}\,\delta_{\text{eq}}\psi^{\bar{\imath}} - \overline{\psi}_{\bar{\imath}}\,\delta_{\text{eq}}{}^2\psi^{\bar{\imath}}\Big]$$

quadratic in bosonic fluctuations

quadratic in fermionic fluctuations

This leads to a Gaussian integral and thus to a superdeterminant that can in principle be calculated (for instance, by fix-point formulae). Because the localization manifold is bosonic, this leads to a ratio of two determinants related to the bosonic and the fermonic fluctuations, respectively.

The dependence on the metric $\sqrt{\mathring{g}}$ has cancelled.

Conclusions

We have presented a general frameworkf for applying supersymmetric localization in supergravity.

One condition that has to be satisfied is that the supersymmetry algebra must close off-shell, i.e. without the need for imposing the equations of motion. This ensures a nilpotent BRST *symmetry and therefore a consistent equivariant cohomology.*

There may be ways of circumventing this requirement by including multi-ghost interactions, but it may make matters considerably more complicated

The only other conditions refer to localization itself. The theory in question must in principle be amenable to localization.